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Mechanical response of a free piezoelectric plate

By D. K. Sinha

Department of Mathematics, Jadavpur University, Calcutta-32

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The paper uses the constitutive equations of a piezoelectric (monoclinic) material, the equations of electricity and the equations of mechanical motion to determine the mechanical response of a free piezoelectric plate.

1. Introduction

The problem of evaluating responses-mechanical or electrical-in piezoelectric transducers owing to prescribed inputs (mechanical or electrical) is an important electromechanical problem from the standpoint of ultrasonic and acoustic engineering, (Mason 1948, Cady 1946). The studies of such problems have long been undertaken from the point of view of circuit theory, (Mason 1948) and it is only in recent years such problems are being tackled by applying the techniques of mechanics of continuous media and of electricity. The efforts in this direction are due to Redwood (1961), Sinha (1962, 1965, 1968, 1966a, 1967a), Giri (1966b), (1967b), Roy (1967c), Das (1967d). The present paper is an analogous attempt and it seeks to investigate the responses in a free piezoelectric plate (unlike those considered in the papers cited above) characterized by a time-decaying polarization gradient, a fact supported by experiments as referred to in Mason (1948b), and Swann (1950). The electrically excited free plate, as observed by Stuetzer (1967), is a useful arrangement for the dynamic measurements of material parameters. The paper makes use of the methods of Laplace transform to facilitate the solution of the problem.

2. Problem, Fundamental Equations and Boundary Conditions.

As our object is to obtain the mechanical response owing to an electrical input, we shall first evolve a relation connecting the two variables representing the two fields-mechanical and electrical. The constitutive relations for the piezoelectric material (for a monoclinic crystal) in the direction of the x-axis are given by

$$T_1 = c_{11}\, S_1 + e_{11}\, E_1 \qquad \text{.. (1)}$$

$$P_1 = e_{11}\, S_1 + K_{11}\, E_1 \qquad \text{.. (2)}$$

where T_1, S_1, E_1 and P_1 are the components of stress, strain, electric intensity and polarization, respectively, and c_{11}, e_{11}, K_{11} are the elastic, piezoelectric and dielectric susceptibility co-efficients.

If ξ be the mechanical displacement in the x-direction, the equation of motion is given by

$$\frac{\partial T_1}{\partial x} = \rho \frac{\partial^2 \xi}{\partial t^2} \qquad \ldots(3)$$

ρ being the density of the material. Since $S_1 = \frac{\partial \xi}{\partial x}$ we have from (1), (2), (3)

$$\rho \frac{\partial^2 \xi}{\partial t^2} = \left(c_{11} - \frac{e_{11}^2}{K_{11}} \right) \frac{\partial^2 \xi}{\partial x^2} - \frac{e_{11}}{K_{11}} \frac{\partial P_1}{\partial x} \qquad \ldots(4)$$

As assumed earlier, we take the polarization gradient $\frac{\partial P_1}{\partial x}$ to be equal to $P_0 e^{-\alpha t}$ where P_0 is a constant and $\alpha > 0$. Substituting this value of $\frac{\partial P_1}{\partial x}$ and taking the Laplace transform (of parameter p) of both sides of the resulting equation, we get

$$\frac{\partial^2 \bar{\xi}}{\partial x^2} - \frac{p^2}{v^2} \bar{\xi} = - \frac{P_0 e_{11}}{(c_{11} K_{11} - e_{11}^2)} \cdot \frac{1}{p + \alpha} \qquad \ldots(5)$$

where $v^2 = \left(c_{11} - \frac{e_{11}^2}{K_{11}} \right) / \rho$

The equation (4) with the assumed value of $\frac{\partial P_1}{\partial x}$ is the fundamental equation of the problem and it is to be solved subject to the condition that the extremities $x = 0$ and $x = X$ to be stress-free *i.e.* $T(0) = 0$ and $T(X) = 0$ Solving (5), we have

$$\bar{\xi} = A e^{\frac{px}{v}} + B e^{-\frac{px}{v}} + \frac{P_0 e_{11}}{\rho} \cdot \frac{1}{p^2 (p + \alpha)} \qquad \ldots(6)$$

Using (5) and (6) in the boundary conditions as well as the relevant expression for T (from which E_{11} has been eliminated by means of (1) and (2)) we get

$$\left. \begin{aligned} A - B &= 0 \\ A e^{\frac{px}{v}} - B e^{-\frac{px}{v}} + \frac{P_0 e_{11} K_{11} v X}{(c_{11} K_{11} - e_{11}^2) p (p + \alpha)} &= 0 \end{aligned} \right\} \qquad \ldots(7)$$

Solving for A and B and substituting in (6) we find $\bar{\xi}$. This, when inversely transformed, gives ξ for any value of x and t. In particular,

$$\begin{aligned} (\bar{\xi})_{x=0} &= A + B - \frac{P_0 e_{11}}{\rho} \cdot \frac{1}{p^2 (p + \alpha)} \\ &= - \frac{2 P_0 e_{11} K_{11} v X}{\left(c_{11} K_{11} - e_{11}^2 \right) p \left(p + \alpha \right) \left(e^{-\frac{pX}{v}} - e^{\frac{pX}{v}} \right)} \\ &\quad + \frac{P_0 e_{11}}{\rho} \cdot \frac{1}{p^2 (p + \alpha)} \end{aligned}$$

Expanding the first term on the righthand side binomially, as in Redwood (1961), and retaining the first order terms we have

$$(\bar{\xi})_{x=0} = -\frac{2\,P_0\,e_{11}\,K_{11}\,v \times e^{-\frac{px}{v}}}{(c_{11}\,K_{11}\,\rho^2{}_{11})\,p\,(p+\alpha)} - \frac{P_0\,e_{11}}{\rho} \cdot \frac{1}{p^2\,(p+\alpha)}$$

Taking the inverse transform, we have

$$(\xi)_{x=0} = \frac{P_0\,e_{11}}{\rho\alpha^2}\,(1 - e^{-\alpha t} - \alpha t), \quad 0 < t < \frac{X}{v}$$

$$= -\frac{2\,P_0\,e_{11}\,K_{11}\,v\,X}{(c_{11}\,k_{11} - e_{11}{}^2)\,\alpha}\,(1 - e^{-\alpha t})$$

$$+ \frac{P_0\,e_{11}}{\rho\alpha^2}\,(-1 + e^{-\alpha t} + \alpha t),\ t > \frac{X}{v}\,.$$

Thus the response by the plate is partly transient, partly linear in time and partly independent of time, similar to what we observe for a plate with rigidly backed extremities.

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